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A Topos Perspective on State-Vector Reduction

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A preliminary investigation is made of possible applications in quantum theory of the topos formed by the collection of all M-sets, where M is a monoid. Earlier results on topos aspects of quantum theory can be rederived in this way. However, the formalism also suggests a new way of constructing a 'neo-realist' interpretation of quantum theory in which the truth values of propositions are determined by the actions of the monoid of strings of finite projection operators. By these means, a novel topos perspective is gained on the concept of state-vector reduction.

KEY WORDS: Quantum theory; topos; monoid; state-vector reduction.

1. INTRODUCTION

The goal of quantum cosmology is to describe in quantum terms the physical universe in its entirety. As a field of study, quantum cosmology is usually construed as a branch of quantum gravity, although some of its most important questions transcend any particular approach to the latter subject.

In this context, it is noteworthy that all the major approaches to quantum gravity assume more or less the standard quantum formalism, both in regard to its mathematical form and to its interpretative framework. Whether such an assumption is justified is debatable, and I have argued elsewhere that, in particular, the *a priori* assumption of a continuum field of numbers (real or complex) would be problematic in a theory where space and time are not representable by a smooth manifold (Isham, 2003). Indeed, it may well be that the entire quantum formalism is only valid in the atomic and nuclear realms, and that something entirely new is needed at the scale of the Planck length.

Nevertheless, in the present paper I shall assume that the standard mathematical formalism of quantum theory is correct and then ask the recurrent question of whether this formalism can yield an interpretation that lies outside the familiar instrumentalism of the standard approach with its emphasis on measurements made

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by an observer who exists 'outside' the system. That one does not wish to invoke an external observer is easy to understand in quantum cosmology.

A simple realist philosophy would aspire to associate with each state $|\psi\rangle$ a definite value for each physical quantity A; equivalently, to each proposition of the form "$A \in \Delta$" (signifying that the physical quantity A has a value that lies in the range Δ of real numbers) there would be associated a truth value $V^{|\psi\rangle}(A \in \Delta)$ that is either 1 (true) or 0 (false). However, the famous Kochen-Specker theorem (Kochen and Specker, 1967) prohibits the existence of any such valuation, and, for those interested in quantum cosmology, this leads to the major challenge of finding a interpretation of the quantum formalism that is non-instrumentalist but which, nevertheless, does not rest on simple 'true-false' valuations.

One possible response to this challenge is to use topos theory. A topos is a category (so there are objects, and arrows from one object to another) with the special property that, in certain critical respects, it behaves like the category of sets (MacLane and Moerdijk, 1992). In particular, just as normal set theory is intimately associated with Boolean algebra (the 'Venn diagram' algebra of subsets of a set is Boolean) so a topos is associated with a more general algebra connected to the sub-objects of objects in the topos.

Concomitantly, in topos theory, one encounters situations in which propositions can be only 'partly' true. The associated truth values lie in a larger set than $\{0, 1\}$, but still maintain the distributive character of classical logic. More precisely, the truth values in a topos lie in what is known as a 'Heyting algebra', which is a generalisation of the Boolean algebra of classical logic: in particular, a Heyting algebra is distributive. The main difference, however, is that, in a Heyting algebra, the law of excluded middle may no longer hold. In other words, there may be elements, P, of the logic such that $P \vee \neg P < 1$ where, here, '$<$' means 'strictly less than' in the partial ordering associated with the logic. This situation is typical of so-called 'intuitionistic logic' and has been much studied by mathematicians concerned with the formal foundations of their subject. The important thing about a logic of this type is that it forms a genuine deductive system—and, as such, can be used as a foundation for mathematics itself—provided only that proof by contradiction is not allowed.

The notion of a proposition being only 'partly true', seems to fit rather well with the fuzzy picture of reality afforded by quantum theory, and the possibility of seriously applying topos ideas to this subject is very intriguing. One attempt, that places much emphasis on the use of generalised truth values, can be found in a series of papers by the author and collaborators (Isham and Butterfield, 1998; Butterfield and Isham, 1999, 2002; Hamilton *et al.*, 2000; Isham, 2005). The fundamental observation in this approach is that if we have a proposition "$A \in \Delta$" for which[2] $0 < \mathrm{Prob}(A \in \Delta;\ |\psi\rangle) < 1$ then although we cannot say that

[2] The quantity $\mathrm{Prob}(A \in \Delta;\ |\psi\rangle)$ denotes the quantum mechanical probability that the proposition "$A \in \Delta$" is 'true' when the quantum state is $|\psi\rangle$. In the standard instrumentalist interpretation(s) of

"$A \in \Delta$" is either true or false (which would correspond to $\text{Prob}(A \in \Delta;\ |\psi\rangle) = 1$ and $\text{Prob}(A \in \Delta;\ |\psi\rangle) = 0$ respectively), nevertheless this proposition may imply other propositions to which the formalism assigns probability 1, and which therefore *can* be said unequivocally to be true. What is not at all obvious, but is nevertheless the case, is that the collections of all such propositions form a distributive logic, and therefore it is possible to *define* the truth value of the proposition "$A \in \Delta$" to be the set of all propositions P that are implied by "$A \in \Delta$" and which are such that $\text{Prob}(P;\ |\psi\rangle) = 1$.

In detail, there is considerably more to the idea than just this, and in the original paper (Isham and Butterfield, 1998), we began by introducing the notion of *coarse-graining* in which the proposition "$A \in \Delta$" is replaced by the 'coarser' proposition[3] "$f(A) \in f(\Delta)$" for some function[4] $f : \mathbb{R} \to \mathbb{R}$. We then ascribed to "$A \in \Delta$" the truth value[5]

$$V^{|\psi\rangle}(A \in \Delta) := \{f_{\mathcal{A}(\mathcal{H})} : \hat{A} \to \hat{B} \mid \text{Prob}(f(A) \in f(\Delta);\ |\psi\rangle) = 1\}. \qquad (1.1)$$

In this approach, the bounded, self-adjoint operators on $\mathcal{H}$ are viewed as the objects in a category $\mathcal{A}(\mathcal{H})$, and a function $f : \mathbb{R} \to \mathbb{R}$ defines an arrow from $\hat{A}$ to $\hat{B}$ if $\hat{B} = f(\hat{A})$. This is the significance of the notation in Eq. (1.1) where the right hand side is to be regarded as a 'sieve'[6] of arrows on the object $\hat{A}$ in the category $\mathcal{A}(\mathcal{H})$. One of the fundamental results in topos theory is that, in any category, the collection of sieves on an object form a Heyting algebra, and hence Eq. (1.1) assigns (contextualised) multi-valued truth values in quantum theory. The actual topos in this example is given by the collection of presheaves[7] over the category $\mathcal{A}(\mathcal{H})$.

quantum theory, the proposition being 'true' means that if a measurement is made of the physical quantity A then the result will definitely be found to lie in $\Delta \subset \mathbb{R}$. For a normalised state $|\psi\rangle$ we have that $\text{Prob}(A \in \Delta;\ |\psi\rangle) = \langle\psi|\,\hat{E}[A \in \Delta]\,|\psi\rangle$ where $\hat{E}[A \in \Delta]$ is the spectral projector onto the eigenspace of $\hat{A}$ associated with eigenvalues that lie in $\Delta \subset \mathbb{R}$.

[3] The key point here is that the proposition "$A \in \Delta$" implies the proposition "$f(A) \in f(\Delta)$" although the converse is generally false. For example, if a physical quantity A has the value 2 then this implies that the value of A^2 is 4. On the other hand, from the knowledge that $A^2 = 4$ we can deduce only that $A = 2$ or -2.

[4] In normal set theory, the notation $f : X \to Y$ means that f is a function from the set X to the set Y. In a general category, the notation $f : X \to Y$ will denote an arrow/morphism whose domain is the object X and whose range is the object Y.

[5] In general, the notation $A := B$ means that the quantity A is *defined* by the expression B. This is frequently of the form stating that A is the set of entities that possesses a particular property, as in the example of Eq. (1.1).

[6] A collection S of arrows with domain O is said to be a 'sieve on O' if for any $f \in S$, $h \circ f \in S$ for all arrows h that can be combined with f (i.e., are which are such that the domain of h is equal to the range of f). Thus a sieve is like a *left ideal* I in a monoid M since $nm \in I$ for all $n \in M$ and $m \in I$. This is one way of understanding why left ideals in monoids are important in topos theory: something that is much exploited in the current paper.

[7] A 'presheaf' $\mathbf{F}$ over a category $\mathcal{C}$ is defined to be (i) to each object A in $\mathcal{C}$, an assignment of a set $\mathbf{F}(A)$; and (ii) to each arrow $f : A \to B$ in $\mathcal{C}$, an assignment of a map $\mathbf{F}(f) : \mathbf{F}(A) \to \mathbf{F}(B)$ such that if $f : A \to B$ and $g : B \to C$ then $\mathbf{F}(g \circ f) : \mathbf{F}(A) \to \mathbf{F}(C)$ satisfies $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$. It is

From a mathematical perspective this structure is correct, nevertheless the underlying theory—of presheaves and the logic of sieves—is not the easiest thing to grasp. So it is natural to wonder if there might be a mathematically simpler way to use topos theory in quantum physics. For example, one can rewrite Eq. (1.1) as

$$V^{|\psi\rangle}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{A} \to \hat{B} \mid \mathrm{Prob}(f(A) \in f(\Delta); |\psi\rangle) = 1\} \tag{1.2}$$

$$= \{f : \mathbb{R} \to \mathbb{R} \mid \hat{E}[f(A) \in f(\Delta)]\,|\psi\rangle = |\psi\rangle\} \tag{1.3}$$

where, in general, $\hat{E}[B \in \Gamma]$ denotes the spectral projector onto the eigenspace of the (bounded, self-adjoint) operator $\hat{B}$ associated with eigenvalues that lie in the range $\Gamma \subset \mathbb{R}$.

By rewriting Eq. (1.1) in the form Eq. (1.3) nothing is lost, and yet Eq. (1.3) looks simpler since it deals directly with functions $f : \mathbb{R} \to \mathbb{R}$, rather than with the arrows that they induce in the category $\mathcal{A}(\mathcal{H})$. In this respect, a key observation is that the right hand side of Eq. (1.3) is actually a *left ideal*[8] in the monoid of functions from $\mathbb{R}$ to $\mathbb{R}$. A left ideal is much like a sieve of arrows (c.f. footnote 6) and yet, arguably, is easier to grasp intuitively.

The present paper takes its cue from replacing Eq. (1.2) with Eq. (1.3), and is grounded in an attempt to exploit the topos structure associated with any monoid, not least because in text books on topos theory this example is invariably introduced early on, and it is a relatively easy one with which to work.

We recall that a monoid is a semi-group with an identity, and thus differs from a group in that inverses of elements may not exist. One obvious example of a monoid is the set of all $n \times n$ matrices in which the combination law is matrix multiplication; the identity is then just the unit matrix. Another basic example of a monoid is the collection, $\mathrm{Map}(X, X)$, of all functions $f : X \to X$ from some set X to itself, with the combination $f \star g$ of a pair f, g of such functions being defined as their composition: $f \star g(x) := f(g(x))$ for all $x \in X$. The monoid identity is just the identity function $\mathrm{id}_X : X \to X$.

For any given monoid M, a key concept is that of a (left) 'M-set'. This is defined to be a set X together with an association to each $m \in M$ of a map $\ell_m : X \to X$ such that (i) if 1 denotes the unit of M then $\ell_1(x) = x$ for all $x \in X$; and (ii) for all $m, n \in M$, we have

$$\ell_m \circ \ell_n = \ell_{mn} \tag{1.4}$$

For simplicity, the element $\ell_m(x) \in X$ will usually be written as mx, and then Eq. (1.4) reads

$$m(nx) = (mn)x \tag{1.5}$$

also required that if $1_A : A \to A$ is the identity arrow at any object A in $\mathcal{C}$, then $\mathbf{F}(1_A) : \mathbf{F}(A) \to \mathbf{F}(A)$ is the identity map.

[8] Recall that a subset $I \subset M$ is a 'left ideal' if $mI := \{mn \in M \mid n \in I\} \subset I$ for all $m \in M$.

for all $x \in X$.

As theoretical physicists, we are very familiar with M-sets for the special case when M is a group: for example, any linear representation of a group is an M-set, as is the action of a group on a manifold in the theory of non-linear group realisations. Indeed, Eq. (1.4) describes a 'realisation' of the monoid M in the monoid, Map(X, X), of all functions of X to itself; as such it can be viewed as a significant generalisation of the idea of a non-linear realisation of a group. As we shall see in the present paper, there are potential physical roles for M-sets in situations where M is definitely *not* a group.

The relation to topos theory becomes clear with the observation that, for any given monoid M, the collection of all M-sets can be given the structure of a topos. The objects in this category are the M-sets themselves, and the arrows/morphisms between a pair of M-sets are the equivariant[9] functions between them. A crucial object in any topos is the 'object of truth values', Ω, which plays the analogue of the set $\{0, 1\}$ in the category of sets. In the case of the topos of M-sets, Ω turns out to be the set of left ideals in M. The close resemblance of a left ideal to a sieve of arrows suggests that it might be possible to recover our earlier results using M-sets rather than the more complicated mathematics of presheaves. This is indeed the case but, as we will see, using the theory of M-sets it is also possible to obtain quite new ideas about generalised quantum valuations.

The basic mathematics of the theory of M-sets is described in Section 2.1. This is applied in Section 2.2 to recover the topos ideas in classical physics that were first discussed by Jeremy Butterfield and myself in Butterfield and Isham (1999). Then, in Section 2.3, we show how topos monoid ideas can be used to recover in a new guise our earlier results on quantum theory as encapsulated in Eq. (1.1). The monoid used in this example is that given by the collection of all bounded, measurable functions from $\mathbb{R}$ to $\mathbb{R}$.

Then, in Section 3 we strike out in a new direction by considering possible roles for the monoid of all bounded operators on the Hilbert space of the quantum theory. In turn, this leads us to consider the monoid consisting of finite strings of projection operators and hence, finally, to a new topos perspective on the familiar, albeit controversial, process of state vector reduction.

2. MONOID ACTIONS AND GENERALISED TRUTH VALUES

2.1. The General Theory

Following standard practice, we denote by BM the category whose objects are (left) M-sets, and whose arrows are M-equivariant maps. Thus, if X and Y are

[9] A function $f : X \to Y$ between M-sets X and Y is *equivariant* if $f(mx) = mf(x)$ for all $m \in M$, $x \in X$.

M-sets, an arrow $f : X \to Y$ in the category BM is a map $f : X \to Y$ such that $f(mx) = mf(x)$ for all $m \in M, x \in X$.

In any topos a key role is played by the 'truth object' Ω. This object has the property that the sub-objects of any object X are in one-to-one correspondence with arrows[10] $\chi : X \to \Omega$. For the category BM, the truth object is the set LM of all left ideals in the monoid M. The action of M on LM is Goldblatt (1984)

$$\ell_m(I) := \{m' \in M \mid m'm \in I\} \tag{2.2}$$

for all $m \in M$. It is immediately clear that the right hand side of Eq. (2.2) is indeed a left ideal in M, and one verifies trivially that Eq. (1.4) (or, equivalently, Eq. (1.5)) is satisfied. Note that for the ideal $1 := M$ we have $\ell_m(1) = 1$ for all $m \in M$. For the ideal $0 := \emptyset$, we have $\ell_m(0) = 0$ for all $m \in M$.

The Heyting algebra structure on LM is defined as follows. The logical 'and' and 'or' operations are $I \wedge J := I \cap J$ and $I \vee J := I \cup J$ respectively, and the unit element and zero element in the algebra are $1 := M$ and $0 := \emptyset$ respectively. The partial order is defined by saying that $I \prec J$ if and only if $I \subseteq J$, and the logical implication $I \Rightarrow J$ is defined by Goldblatt (1984)

$$I \Rightarrow J := \{m \in M \mid \ell_m(I) \subset \ell_m(J)\}. \tag{2.3}$$

As in all Heyting algebras, $\neg I$ is defined by $\neg I := I \Rightarrow 0$; thus, in BM,

$$\neg I := \{m \in M \mid \forall n, nm \notin I\}. \tag{2.4}$$

Our task, then, is to seek physical applications for truth values that lie in the Heyting algebra of all left ideals in a monoid. From the perspective of topos theory, the natural way of finding such truth values arises from the fundamental nature of sub-objects: namely, the existence of a one-to-one correspondence between sub-objects of an object X and arrows from X to Ω. In the case of a topos BM, the sub-objects of an object X in BM are the M-invariant subsets of X, where a subset Y of X is said to be 'M-invariant' if for all $m \in M$ and $y \in Y$ we have $my \in Y$. Then, a BM-arrow $\chi : X \to LM$ (i.e., χ is an M-equivariant function from X to LM) determines the subset

$$J^\chi \subset X := \{x \in X \mid \chi(x) = 1\} \tag{2.5}$$

which, as can readily be checked, is indeed M-invariant. Conversely, if $J \subset X$ is an M-invariant subset of X, then the associated 'characteristic arrow' $\chi^J : X \to LM$

[10] For the category of sets, Ω is just the set $\{0, 1\}$. If J is a subset of the set X then the associated characteristic map $\chi^J : X \to \{0, 1\}$ is

$$\chi^J(x) := \begin{cases} 1 & \text{if } x \in J; \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

is defined by

$$\chi^J(x) := \{m \in M \mid mx \in J\}. \tag{2.6}$$

It is easy to see that, since J is M-invariant, the right hand side of Eq. (2.6) is indeed a left-ideal in M, and hence an element of LM.

One can think of the right hand side of Eq. (2.6) as being a measure of the 'extent' to which x is an element of J: the more elements of M send x into J (i.e., the larger the right hand side of Eq. (2.6)) the 'closer' x is to being in J. With this in mind, we rewrite Eq. (2.6) as

$$[x \in J]_{BM} := \{m \in M \mid mx \in J\} \tag{2.7}$$

and view Eq. (2.7) as the truth value in the topos BM for the proposition "$x \in J$". Note that if x belongs to J then $[x \in J]_{\mathrm{BM}} = M$—the unit element of the Heyting algebra LM.

In practice, we shall use a slight generalisation of the example of Eq. (2.7). Namely, if X is an M-set let $\mathbf{K} := \{K_m, m \in M\}$ be a family of subsets of X that satisfy the conditions, for all m,[11]

$$m'K_m \subset K_{m'm} \tag{2.8}$$

for all $m' \in M$.[12] Then if we define (cf. Eq. (2.7))

$$[x \in \mathbf{K}]_{BM} := \{m \in M \mid mx \in K_m\} \tag{2.9}$$

it is easy to check that the right hand side of Eq. (2.9) is a left ideal in M. Thus another structure that can give a source of generalised truth values is a family of subsets $\{K_m \subset X, m \in M\}$ that satisfies Eq. (2.8).[13]

In particular, if K is *any* subset of X (not necessarily M-invariant) and if we define $K_m := mK$, we see at once that Eq. (2.8) is satisfied. In short, any subset

[11] If K is any subset of the M-set X, we denote by mK the set $\{mx \mid x \in K\}$.

[12] On the face of it, we could also consider families of sets of the form $\mathbf{K}_I := \{K_m \mid m \in I\}$ for any ideal I in M, since Eq. (2.8) still makes sense in this case. However, we can reduce this to the case with $I := M$ by choosing K_m to be the empty set for all $m \notin I$.

[13] With some effort it can be shown that families $\{K_m, m \in M\}$ satisfying Eq. (2.8) are in one-to-one correspondence with equivariant maps $\lambda : X \times M \to LM$. Specifically, given such a map λ define $K_m^\lambda := \{x \in X \mid \lambda(x, m) = 1\}$. Conversely, given a family $\mathbf{K} = \{K_m, m \in M\}$ satisfying Eq. (2.8) define $\lambda^{\mathbf{K}}(x, m) := \{m' \in M \mid m'x \in K_{m'm}\}$. The significance of this result is that equivariant maps $\lambda : X \times M \to LM$ correspond to the points (in the ordinary set-theoretic sense) of the power object PX of the object X in BM (Goldblatt, 1984). This is an important part of the general theory of the topos BM but it has been relegated to a footnote since I am trying to minimise the amount of 'heavy' mathematics in the main text.

$K \subset X$ gives rise to a generalised truth value[14]

$$[x \in K]_{\mathrm{BM}} := \{m \in M \mid mx \in mK\}. \tag{2.10}$$

It can readily be checked that the right hand side of Eq. (2.10) is indeed a left ideal in the monoid M. This example will play a central role in the applications to quantum theory.

More generally, if K_1, K_2 are any pair of subsets of X we can define

$$[K_1 \subset K_2]_{BM} := \{m \in M \mid mK_1 \subset mK_2\}. \tag{2.11}$$

A particular example of Eq. (2.10) is $K := \{y\}$ for some $y \in X$. In this special case, Eq. (2.10) can be written as

$$[x = y]_{BM} := \{m \in M \mid mx = my\}. \tag{2.12}$$

The right hand side of Eq. (2.12) is clearly a left ideal in M: for if $m \in M$ is such that $mx = my$ then, trivially, $nmx = nmy$ for all $n \in M$. Thus Eq. (2.12) is a measure in the topos of M-sets of the extent to which the points x, y in X are 'partially equal'. Indeed, $[x = y]_{BM}$ is larger the 'closer' x and y are to being equal, with $[x = y]_{BM} = M$ (the identity of the Heyting algebra LM) if $x = y$.

2.2. A Monoid Concept of 'Nearness to Truth' in Classical Physics

An application of a topos of type BM arises in classical physics. Here we have a classical state space $\mathcal{S}$ (a smooth manifold) in which each physical quantity A is represented by a smooth, real-valued function, $\overline{A}$, on $\mathcal{S}$. Each state $s \in \mathcal{S}$ gives rise to a simple valuation on propositions of the form

$$V^s(A \in \Delta) := \begin{cases} 1, & \text{if } \overline{A}(s) \in \Delta; \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

In other words, the proposition "$A \in \Delta$" is true if the state s is such that $\overline{A}(s)$ belongs to Δ; otherwise it is false. Equivalently, "$A \in \Delta$" is true if and only if $s \in \overline{A}^{-1}(\Delta) := \{s \in \mathcal{S} \mid \overline{A}(s) \in \Delta\}$.

Such a simple 'either-or' perspective seems natural in the context of classical physics, and indeed one may wonder what else the proposition "$A \in \Delta$" could mean other than the information conveyed by Eq. (2.13). All this seems clear-cut—but is it really so? For suppose s is a state that does not belong to $\overline{A}^{-1}(\Delta)$

[14] One must be careful not to confuse Eq. (2.10) with Eq. (2.7). If K is an M-invariant subset of X, the definition in Eq. (2.10) still makes sense, but this is generally not the same as Eq. (2.7) since there will typically be elements $m \in M$ such that mK is a proper subset of K. When K is an invariant subset we will use Eq. (2.7) (rather than Eq. (2.10)) since this corresponds to thinking of K as a sub-object of X in BM.

but which, nevertheless, is 'almost' in this subset (so that $\overline{A}(s)$ 'almost' belongs to Δ): is there not then some sense in which the proposition "$A \in \Delta$" is 'almost true'? Contrariwise, suppose s is such that $\overline{A}(s)$ belongs to Δ, but only just so (i.e., $\overline{A}(s)$ is 'close' to the edges of Δ): then is not "$A \in \Delta$" 'almost false', or 'only just true'? Such grey-scale judgements are made frequently in daily life, but at first sight there seems to be no role for them in the harsh, black-and-white mathematics of classical physics.

From a mathematical perspective, the problem is how to judge the nearness of any point s in $\mathcal{S}$ to the subset $\overline{A}^{-1}(\Delta)$ of $\mathcal{S}$. Of course, we could always put a metric on $\mathcal{S}$, but there is in general no obvious or natural way of choosing this (notwithstanding the fact that, in classical physics, $\mathcal{S}$ is a symplectic manifold with a canonical two-form).

However, a more appealing approach is based on the observation that if the state s is such that $\overline{A}(s) \in \Delta$ then, of necessity, $f(\overline{A}(s)) \in f(\Delta)$ for any smooth function $f : \mathbb{R} \to \mathbb{R}$. This type of coarse-graining was discussed in detail in Butterfield and Isham (1999) in the context of assigning truth values to propositions "$A \in \Delta$" when the state of the system is a macrostate $M \subset \mathcal{S}$. In the present case, we have $M = \{s\}$, and then the analysis in Butterfield and Isham (1999) results in the generalised valuation[15]

$$V^s(A \in \Delta) := \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f(\overline{A}(s)) \in f(\Delta)\} \tag{2.14}$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ denotes the set of smooth (i.e., infinitely differentiable) functions $f : \mathbb{R} \to \mathbb{R}$.

In Butterfield and Isham (1999), the discussion of Eq. (2.14) employed a topos of presheaves with truth values being sieves. However, Eq. (2.14) can easily be reinterpreted in terms of a topos BM. Specifically, we note that, since the composition of a pair of smooth functions is itself smooth, the set $C^\infty(\mathbb{R}, \mathbb{R})$ can be given a monoid structure whose combination law is defined as $f \star g\,(r) := f(g(r))$ for all $r \in \mathbb{R}$. We then see at once that the right hand side of Eq. (2.14) is actually a *left ideal* in this monoid. Indeed, if $f \in C^\infty(\mathbb{R}, \mathbb{R})$ is such that $f(\overline{A}(s)) \in f(\Delta)$ then, trivially, for all $h : \mathbb{R} \to \mathbb{R}$ we have $h(f(\overline{A}(s))) \in h(f(\Delta))$. Thus $f(\overline{A}(s)) \in f(\Delta)$ implies that, for all $h \in C^\infty(\mathbb{R}, \mathbb{R})$, we have $h \star f\,(\overline{A}(s)) \in h \star f\,(\Delta)$, which means precisely that the right hand side of Eq. (2.14) is a left ideal in the monoid $C^\infty(\mathbb{R}, \mathbb{R})$.

This remark suggests that the generalised valuation in Eq. (2.14) could be understood in terms of the topos of $C^\infty(\mathbb{R}, \mathbb{R})$-sets. This is indeed the case: in particular, we consider the obvious action of the monoid $C^\infty(\mathbb{R}, \mathbb{R})$ on the set $\mathbb{R}$,

[15] The coarse-graining of the original proposition "$A \in \Delta$" that is implicit in Eq. (2.14) can be seen by noting that $f(\overline{A}(s)) \in f(\Delta)$ if and only if $\overline{A}(s) \in f^{-1}(f(\Delta))$, and hence Eq. (2.14) assigns to the proposition "$A \in \Delta$" all those weaker (coarse-grained) propositions "$A \in f^{-1}(f(\Delta))$" which are 'true' in the normal sense of the word.

defined by[16]

$$\ell_f(r) := f(r) \tag{2.15}$$

for all $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $r \in \mathbb{R}$. Now, for each fixed state s in $\mathcal{S}$, $\overline{A}(s)$ belongs to $\mathbb{R}$, and hence, applying Eq. (2.10) with $X := \mathbb{R}$, $x := \overline{A}(s)$, and $K := \Delta \subset \mathbb{R}$, we see that

$$[\overline{A}(s) \in \Delta]_{\mathrm{BC}^\infty(\mathbb{R},\mathbb{R})} = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f(\overline{A}(s)) \in f(\Delta)\}. \tag{2.16}$$

In other words, the generalised valuation in Eq. (2.14) is just $[\overline{A}(s) \in \Delta]_{\mathrm{BC}^\infty(\mathbb{R},\mathbb{R})}$.

With an eye to the application to quantum theory to be discussed in Sec. 2.3, we note that another monoid interpretation of Eq. (2.14) can be obtained by considering the action of the monoid $C^\infty(\mathbb{R}, \mathbb{R})$ on the set $C^\infty(\mathcal{S}, \mathbb{R})$ whose elements (smooth, real-valued functions $\overline{A}$, $\overline{B}$ on $\mathcal{S}$) represent physical quantities in the system. Specifically, we define

$$\ell_f(\overline{B}) := f \circ \overline{B} \tag{2.17}$$

for all $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\overline{B} \in C^\infty(\mathcal{S}, \mathbb{R})$. We can also define an action of $C^\infty(\mathbb{R}, \mathbb{R})$ on the family, $P(\mathbb{R})$, of subsets of $\mathbb{R}$ by

$$\ell_f(\Gamma) := f(\Gamma) \tag{2.18}$$

for all $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\Gamma \subset \mathbb{R}$. These operations combine to give an action of the monoid $C^\infty(\mathbb{R}, \mathbb{R})$ on $C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R})$ defined by

$$\begin{aligned} \ell_f : C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R}) &\to C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R}) \\ (\overline{B}, \Gamma) &\mapsto (f \circ \overline{B}, f(\Gamma)) \end{aligned} \tag{2.19}$$

If desired, this can also be viewed as defining an action of $C^\infty(\mathbb{R}, \mathbb{R})$ on the space of propositions of the type "$B \in \Gamma$". In other words, the proposition "$B \in \Gamma$" is mapped by f to the proposition "$f(B) \in f(\Gamma)$".

We then define, for each state $s \in \mathcal{S}$, the set

$$E^s := \{(\overline{B}, \Gamma) \mid \overline{B}(s) \in \Gamma\} \tag{2.20}$$

and note that this subset of $C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R})$ is *invariant* under the action of the monoid $C^\infty(\mathbb{R}, \mathbb{R})$ (for, if $\overline{B}(s) \in \Gamma$ then certainly $f(B(s)) \in f(\Gamma)$ for

[16] This is a special case of a much wider class of examples. Indeed, for any set X there is a natural action of the monoid $\mathrm{Map}(X, X)$ on X given by (cf. Eq. (2.15)) $\ell_f(x) := f(x)$ for all $f \in \mathrm{Map}(X, X)$ and $x \in X$. If X is a topological space, it is natural to restrict attention to the sub-monoid $C(X, X)$ of continuous functions from X to X. If X is a differentiable manifold, one would use the sub-monoid $C^\infty(X, X)$ of smooth functions from X to X. Note that these subsets of $\mathrm{Map}(X, X)$ are indeed sub-*monoids* since the composition of a pair of continuous (resp. smooth) functions is itself continuous (resp. smooth). More generally, if X is an object in an arbitrary (small) category with a terminal object 1, one could use the monoid $\mathrm{Hom}(X, X)$ of arrows whose domain and range is X, and with the obvious action on the global elements $x : 1 \to X$ in which $f \in \mathrm{Hom}(X, X)$ transforms x to $f \circ x$.

all $f \in C^\infty(\mathbb{R}, \mathbb{R})$). As such, it is a sub-object of $C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R})$ in the topos $BC^\infty(\mathbb{R}, \mathbb{R})$, and hence there is an associated characteristic arrow from $C^\infty(\mathcal{S}, \mathbb{R}) \times P(\mathbb{R})$ to the set $LC^\infty(\mathbb{R}, \mathbb{R})$ of left ideals of $C^\infty(\mathbb{R}, \mathbb{R})$. According to the general result in Eq. (2.7), this gives rise to the generalised truth value

$$[(\overline{A}, \Delta) \in E^s]_{\mathrm{BC}^\infty(\mathbb{R},\mathbb{R})} = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid (f \circ \overline{A}, f(\Delta)) \in E^s\}$$
$$= \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f(\overline{A}(s)) \in f(\Delta)\} \qquad (2.21)$$

which is precisely the right hand side of the generalised valuation Eq. (2.14).

2.3. Using the Monoid $M(\mathbb{R}, \mathbb{R})$ in Quantum Theory

We can now discuss a monoid reinterpretation of the generalised valuation Eq. (2.13)[17]

$$V^{|\psi\rangle}(A \in \Delta) := \{f : \mathbb{R} \to \mathbb{R} \mid \hat{E}[f(A) \in f(\Delta)]\,|\psi\rangle = |\psi\rangle\} \qquad (2.22)$$

that was introduced in Isham and Butterfield (1998) in the context of our topos analysis of the Kochen-Specher theorem. In that earlier[18] paper, the right hand side of Eq. (2.22) was interpreted as a sieve of arrows on the object $\hat{A}$ in a category[19] $\mathcal{A}(\mathcal{H})$ whose objects are bounded, self-adjoint operators, and whose arrows $f_{\mathcal{A}(\mathcal{H})} : \hat{A} \to \hat{B}$ are defined to be all real functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $\hat{B} = f(\hat{A})$.

The underlying mathematics is, again, presheaf theory, but in the light of the discussion above, it is reasonable to enquire if Eq. (2.22) can be reinterpreted in a monoid language. To this end, first recall that if $\hat{A}$ is any bounded, self-adjoint operator then, for any bounded, measurable function $f : \mathbb{R} \to \mathbb{R}$, the operator $f(\hat{A})$ can be defined using the spectral theorem for $\hat{A}$, and this operator is also bounded and self-adjoint. We denote the set of all such functions $f : \mathbb{R} \to \mathbb{R}$ by $M(\mathbb{R}, \mathbb{R})$, and note that this can be given a monoid structure by composition since the composition of any pair of bounded and measurable functions is itself bounded and measurable.

Then the critical observation is that the right hand side of Eq. (2.22) is actually a *left ideal* in this monoid. The reason is analogous to that in Section 2.2 in regard to the discussion following Eq. (2.14). Specifically, for any $h \in M(\mathbb{R}, \mathbb{R})$, we

[17] Note that the right hand side of Eq. (2.22) is invariant under the scaling $|\psi\rangle \mapsto \lambda\,|\psi\rangle$ for all non-zero complex numbers λ. Hence Eq. (2.22) defines a valuation on the projective Hilbert space $P\mathcal{H}$ of all rays in $\mathcal{H}$, and we could just as well denote the left hand side as $V^{[\,|\psi\rangle]}(A \in \Delta)$ where $[\,|\psi\rangle]$ denotes the ray that passes through the vector $|\psi\rangle$.

[18] See Döring (2005) for a recent, and very sophisticated, analysis of the Kochen-Specher theorem using the mathematics of presheaves.

[19] In Isham and Butterfield (1998) the category $\mathcal{A}(\mathcal{H})$ was denoted $\mathcal{O}$.

have[20]

$$\hat{E}[B \in \Gamma] \preceq E[h(B) \in h(\Gamma)] \tag{2.23}$$

in the partial ordering of the lattice of projection operators. It follows at once that if $|\psi\rangle$ and f are such that $\hat{E}[f(A) \in f(\Delta)]\,|\psi\rangle = |\psi\rangle$ then $\hat{E}[h(f(A)) \in h(f(\Delta))]\,|\psi\rangle = |\psi\rangle$ for all $h \in M(\mathbb{R}, \mathbb{R})$. But this is precisely the statement that Eq. (2.22) is a left ideal in the monoid $M(\mathbb{R}, \mathbb{R})$.

This suggests strongly that the generalised valuation Eq. (2.22) can be reinterpreted using the language of the topos of $M(\mathbb{R}, \mathbb{R})$-sets. To complete this identification it is necessary to find an appropriate set on which the monoid $M(\mathbb{R}, \mathbb{R})$ acts, and then apply the general result in Eq. (2.7).

The first relevant observation is that if $\mathcal{A}(\mathcal{H})$ denotes the set of all bounded, self-adjoint operators on $\mathcal{H}$, then the operation whereby $\hat{B} \in \mathcal{A}(\mathcal{H})$ is replaced by $f(\hat{B})$, with $f \in M(\mathbb{R}, \mathbb{R})$, can be viewed as a left action of the monoid $M(\mathbb{R}, \mathbb{R})$ on $\mathcal{A}(\mathcal{H})$ (cf. Equation (2.17)). Similarly, if $B(\mathbb{R})$ denotes the collection of bounded, Borel subsets of $\mathbb{R}$, then an action of $M(\mathbb{R}, \mathbb{R})$ on $B(\mathbb{R})$ can be defined[21] by letting $f \in M(\mathbb{R}, \mathbb{R})$ take $\Gamma \in B(\mathbb{R})$ to $f(\Gamma)$. This is a direct analogue of the action, Eq. (2.18), in the classical case.

Combining these two operations gives a left action of the monoid $M(\mathbb{R}, \mathbb{R})$ on $\mathcal{A}(\mathcal{H}) \times B(\mathbb{R})$ which is defined as (cf. Eq. (2.19))

$$\begin{aligned} \ell_f : \mathcal{A}(\mathcal{H}) \times B(\mathbb{R}) &\to \mathcal{A}(\mathcal{H}) \times B(\mathbb{R}) \\ (\hat{B}, \Gamma) &\mapsto (f(\hat{B}), f(\Gamma)) \end{aligned} \tag{2.24}$$

for all $f \in M(\mathbb{R}, \mathbb{R})$. This can also be viewed as an action of $M(\mathbb{R}, \mathbb{R})$ on the space of propositions of the form "$B \in \Gamma$". In any event, what is important is that to each vector $|\psi\rangle \in \mathcal{H}$, we can define (cf. Eq. (2.20))

$$E^{|\psi\rangle} := \{(\hat{B}, \Gamma) \mid \hat{E}[B \in \Gamma]\,|\psi\rangle = |\psi\rangle\}. \tag{2.25}$$

Then the crucial observation is that this subset of $\mathcal{A}(\mathcal{H}) \times B(\mathbb{R})$ is *invariant* under the action of the monoid $M(\mathbb{R}, \mathbb{R})$. This follows at once from the partial ordering relation in Eq. (2.23) which guarantees that if $(\hat{B}, \Gamma) \in E^{|\psi\rangle}$ (so that $\hat{E}[B \in \Gamma]\,|\psi\rangle = |\psi\rangle$) then $(f(\hat{B}), f(\Gamma)) \in E^{|\psi\rangle}$ for all $f \in M(\mathbb{R}, \mathbb{R})$.

[20] Strictly speaking, Γ has to be a Borel subset of $\mathbb{R}$ in order for the spectral projector $\hat{E}[B \in \Gamma]$ to exist. However, this then raises the difficulty that if Γ is Borel it is not necessarily the case that $h(\Gamma)$ is Borel for arbitrary $h \in M(\mathbb{R}, \mathbb{R})$. This issue is resolved in Butterfield and Isham (1999) but we will not dwell on it here.

[21] It is necessary to take into account the cautionary remark in footnote 20.

We can now use the general definition in Eq. (2.7) to compute this subset's characteristic function from $\mathcal{A}(\mathcal{H}) \times B(\mathbb{R})$ to $LM(\mathbb{R}, \mathbb{R})$. This gives

$$[(\hat{A}, \Delta) \in E^{|\psi\rangle}]_{BM(\mathbb{R},\mathbb{R})} := \{f \in M(\mathbb{R}, \mathbb{R}) \mid (f(\hat{A}), f(\Delta)) \in E^{|\psi\rangle}\} \tag{2.26}$$

$$= \{f \in M(\mathbb{R}, \mathbb{R}) \mid \hat{E}[f(A) \in f(\Delta)]\,|\psi\rangle = |\psi\rangle\} \tag{2.27}$$

which is precisely the right hand side of Eq. (2.22). Thus the generalised truth value in Eq. (2.22) has an interpretation in terms of the topos of $M(\mathbb{R}, \mathbb{R})$-sets. Note the close analogy with the result Eq. (2.21) of the classical theory.

3. A TOPOS INTERPRETATION OF STATE-VECTOR REDUCTION

3.1. Actions of the Monoid $L(\mathcal{H})$

So far, our application of monoid theory to quantum mechanics has been to use the language of $M(\mathbb{R}, \mathbb{R})$-sets to re-express earlier results obtained originally using presheaf theory. However, we wish now to develop a new, 'neo-realist' interpretation of quantum theory that uses a topos of M-sets in a fundamental way.

Given a quantum theory with a Hilbert space $\mathcal{H}$, one obvious monoid to consider is the set $L(\mathcal{H})$ of all bounded, linear operators on $\mathcal{H}$. The monoid composition law is the operator product, and the unit element is simply the unit operator $\hat{1}$. A related monoid is obtained by defining two operators to be equivalent, $\hat{A} \equiv \hat{B}$, if there exists $\lambda \in \mathbb{C}_*$ (the non-zero complex numbers) such that $\hat{A} = \lambda\hat{B}$. We denote the set of equivalence classes as $L(\mathcal{H})/\mathbb{C}_*$ and note that this can be given a monoid structure with the combination law

$$[\hat{A}][\hat{B}] := [\hat{A}\hat{B}] \tag{3.1}$$

where $[\hat{A}]$ denotes the equivalence class of $\hat{A}$. This particular monoid was discussed in quantum theory many years ago in the context of the theory of Baer $*$-semigroups (Pool, 1975; Beltrametti and Cassinelli, 1981).

The obvious set on which the monoid $L(\mathcal{H})$ acts is $\mathcal{H}$ itself, with $\ell_{\hat{A}}(|\psi\rangle) := \hat{A}\,|\psi\rangle$ for all $\hat{A} \in L(\mathcal{H})$ and $|\psi\rangle \in \mathcal{H}$.

Another natural action is on the projective Hilbert space $P\mathcal{H}$, with the action on any ray $[\,|\psi\rangle]$ (the ray that passes through the (non-null) vector $|\psi\rangle$) being

$$\ell_{\hat{A}}([\,|\psi\rangle]) := [\hat{A}\,|\psi\rangle] \tag{3.2}$$

Here, the meaning of the symbol $[\hat{A}\,|\psi\rangle]$ is as follows. If $\hat{A}\,|\psi\rangle \neq 0$, then $[\hat{A}\,|\psi\rangle]$ denotes the ray that passes through $\hat{A}\,|\psi\rangle$. However, if $\hat{A}\,|\psi\rangle = 0$, then $[\hat{A}\,|\psi\rangle] = [0]$ denotes a special point that must be added to the projective Hilbert space. Thus the action of our monoid is not on $P\mathcal{H}$ but on $P\mathcal{H} \cup [0]$. Of course, $\ell_{\hat{A}}[0] = [0]$ for all operators $\hat{A}$ in the monoid $L(\mathcal{H})$. In other words, $[0]$ is an *absorbing* element for the action of $L(\mathcal{H})$ on $P\mathcal{H} \cup [0]$.

The action on vectors can be extended to give an action of the monoid $L(\mathcal{H})$ on arbitrary closed, linear subspaces of $\mathcal{H}$. Specifically, if $K \subset \mathcal{H}$ is such a subspace then, for all $\hat{A} \in L(\mathcal{H})$, we define

$$\ell_{\hat{A}}(K) := \hat{A}K := \{\hat{A}\,|\psi\rangle \mid |\psi\rangle \in K\}^{\mathrm{cl}} \tag{3.3}$$

where the superscript $\{\,\}^{\mathrm{cl}}$ signifies that the topological closure is to be taken of the quantity inside the parentheses. Note that since there is a one-to-one correspondence between closed, linear subspaces on $\mathcal{H}$ and projection operators, Eq. (3.3) also generates an action of the monoid $L(\mathcal{H})$ on the collection of projectors. However, there is no obvious way of writing down explicitly what $\hat{A}$ does to any particular projector.

From a projective perspective, we denote by PK the set of all rays passing through the non-null vectors in K. We then get an action of $L(\mathcal{H})$ on $PK \cup [0]$ defined by

$$\ell_{\hat{A}}(PK) := \bigcup_{[\,|\psi\rangle]\in PK} [\hat{A}\,|\psi\rangle] \tag{3.4}$$

and with $\ell_{\hat{A}}[0] := 0$ as before.

If one thinks of quantum states as being represented by normalised vectors then one might try to define an action of $L(\mathcal{H})$ by

$$|\psi\rangle \mapsto \frac{\hat{A}\,|\psi\rangle}{\|\hat{A}\,|\psi\rangle\|}. \tag{3.5}$$

Note that the right hand side of Eq. (3.5) is invariant under the transformation $\hat{A} \mapsto \lambda\hat{A}$, $\lambda \in \mathbb{C}_*$. Thus Eq. (3.5) passes to an action of the monoid $L(\mathcal{H})/\mathbb{C}_*$. There is an analogue of Eq. (3.5) on density matrices

$$\hat{\rho} \mapsto \frac{\hat{A}\rho\hat{A}^\dagger}{\operatorname{tr}(\hat{\rho}\hat{A}\hat{A}^\dagger)}. \tag{3.6}$$

We note however that Eq. (3.5) is only defined if $\hat{A}\,|\psi\rangle \neq 0$, and similarly Eq. (3.6) requires $\operatorname{tr}(\hat{\rho}\hat{A}\hat{A}^\dagger) \neq 0$. This means that neither Eq. (3.5) or Eq. (3.6) corresponds to a well-defined action of the monoid $L(\mathcal{H})$: we shall return to this problem later.

3.2. Truth Values Using the Monoid $L(\mathcal{H})$

Let us now consider how truth values in the set of left ideals of $L(\mathcal{H})$ could arise. One of the simplest expressions is Eq. (2.12) which, for the monoid action of $L(\mathcal{H})$ on $\mathcal{H}$, reads

$$[\,|\psi\rangle = |\phi\rangle]_{BL(\mathcal{H})} := \{\hat{B} \in L(\mathcal{H}) \mid \hat{B}\,|\psi\rangle = \hat{B}\,|\phi\rangle\} \tag{3.7}$$

$$= \{\hat{B} \in L(\mathcal{H}) \mid \hat{B}(\,|\psi\rangle - |\phi\rangle) = 0\} \tag{3.8}$$

which is clearly a left ideal of $L(\mathcal{H})$. There is an analogous expression on the extended projective Hilbert space (i.e., on $P\mathcal{H} \cup [0]$) of the form

$$[[\,|\psi\rangle] = [\,|\phi\rangle]]_{BL(\mathcal{H})} := \{\hat{B} \in L(\mathcal{H}) \mid [\hat{B}\,|\psi\rangle] = [\hat{B}\,|\phi\rangle]\,\}. \tag{3.9}$$

Note that the equation $[\hat{B}\,|\psi\rangle] = [\hat{B}\,|\phi\rangle]$ implies that $\hat{B}\,|\psi\rangle = 0$ if, and only if, $\hat{B}\,|\phi\rangle = 0$.

From a mathematical perspective, Eq. (3.8) is an interesting Heyting-algebra valued measure of the extent to which the vectors $|\psi\rangle$ and $|\phi\rangle$ are not equal. However, as it stands, it is hard to give any physical meaning to this expression. Basically, the problem is that the monoid $L(\mathcal{H})$ consists of *all* bounded operators, whereas, in quantum theory, the most important operators are unitary operators and self-adjoint operators.

We could consider the sub-monoid of unitary operators, but this is uninteresting since a unitary operator is invertible, and hence one-to-one. This means that, for example, the analogue of Eq. (3.8) for unitary operators is the empty set unless $|\psi\rangle = |\phi\rangle$.

One might be tempted to consider the collection $\mathcal{A}(\mathcal{H})$ of bounded, self-adjoint operators on $\mathcal{H}$, but this is not a sub-monoid of $L(\mathcal{H})$ since the product of self-adjoint operators is not itself self-adjoint unless they commute. However, this remark suggests another possibility which, it transpires, *is* fruitful: namely, consider the subset, $\mathrm{Pr}\mathcal{A}(\mathcal{H})$, of $L(\mathcal{H})$ consisting of all finite products of self-adjoint operators. This *is* a sub-monoid, and gives rise to the expression

$$\begin{aligned}[\,|\psi\rangle = |\phi\rangle]_{B\mathrm{Pr}\mathcal{A}(\mathcal{H})} := \{\hat{A}_n\hat{A}_{n-1}, \ldots, \hat{A}_1 \mid \hat{A}_n\hat{A}_{n-1}, \ldots, \hat{A}_1\,|\psi\rangle \\ = \hat{A}_n\hat{A}_{n-1}, \ldots, \hat{A}_1\,|\phi\rangle\}.\end{aligned} \tag{3.10}$$

This expression still has no obvious physical meaning, but it does suggest one thing very strongly: namely, the process of state-vector reduction! This is the procedure whereby if a series of (ideal) measurements is made of physical quantities whose corresponding outcomes are represented by the projection operators $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ respectively, then after the measurements are made (neglecting time development between them) the state vector has been reduced to

$$|\psi\rangle \mapsto \hat{P}_n\hat{P}_{n-1}, \ldots, \hat{P}_1\,|\psi\rangle. \tag{3.11}$$

Of course, this can be viewed as the result of a series of reductions

$$|\psi\rangle \mapsto \hat{P}_1\,|\psi\rangle \mapsto \hat{P}_2\hat{P}_1\,|\psi\rangle \mapsto \cdots \mapsto \hat{P}_n\hat{P}_{n-1}, \ldots, \hat{P}_1\,|\psi\rangle. \tag{3.12}$$

Actually, Eq. (3.11) is not quite correct, as we need to consider the normalisation of the reduced vector. For the moment though, we can say that the key idea is to think of the reduction Eq. (3.11) as being the result of an action on $\mathcal{H}$ of the sub-monoid,[22] $\mathrm{Pr}P(\mathcal{H})$, of finite products of projection operators.

For this particular monoid, the general equation Eq. (2.12) reads

$$\begin{aligned}[\,|\psi\rangle = |\phi\rangle]_{B\mathrm{Pr}P(\mathcal{H})} := \{&\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1 \mid \hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle \\ &= \hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\phi\rangle\}\end{aligned} \tag{3.13}$$

or, perhaps better, we should use rays in the Hilbert space and define

$$\begin{aligned}[\,[|\psi\rangle] = [|\phi\rangle]\,]_{B\mathrm{Pr}P(\mathcal{H})} := \{&\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1 \mid [\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle] \\ &= [\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\phi\rangle]\,\}\end{aligned} \tag{3.14}$$

Note that the right hand side of Eq. (3.14) is equivalent to the statement that there exists $\lambda \in \mathbb{C}_*$ such that $\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle = \lambda\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\phi\rangle$.

Unlike Eq. (3.10), the expressions in Eq. (3.13) and Eq. (3.14) *do* have a very interesting physical interpretation. Namely, they assign (in a slightly different way) as a measure of the similarity between two state vectors the collection of those series of ideal measurements which, *if* they were performed, give reduced vectors that can no longer be distinguished from each other.

Strictly speaking, this is not quite correct, and will be amended shortly in Section 3.3. However, before doing that we note that this idea can be developed immediately to attain our goal of producing a new type of truth value for propositions "$A \in \Delta$" in quantum theory. For let $\mathcal{H}_{A\in\Delta}$ denote the subspace of $\mathcal{H}$ that is the image of the spectral projector $\hat{E}[A \in \Delta]$; i.e., the proposition "$A \in \Delta$" is true with probability 1 for all states $|\phi\rangle$ in $\mathcal{H}_{A\in\Delta}$. Then, based on the general result Eq. (2.10), we can define the new generalised valuation

$$\begin{aligned}V^{|\psi\rangle}(A \in \Delta)_{B\mathrm{Pr}P(\mathcal{H})} &:= [\,|\psi\rangle \in \mathcal{H}_{A\in\Delta}]_{B\mathrm{Pr}P(\mathcal{H})} \\ &= \{\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1 \mid \hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle \\ &\qquad \in \hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\mathcal{H}_{A\in\Delta}\}.\end{aligned} \tag{3.15}$$

Alternatively, and probably better in terms of physical meaning, we can adopt the projective perspective and define

$$\begin{aligned}V^{[\,|\psi\rangle]}(A \in \Delta)_{B\mathrm{Pr}P(\mathcal{H})} &:= [\,[|\psi\rangle] \in P\mathcal{H}_{A\in\Delta}]_{B\mathrm{Pr}P(\mathcal{H})} \\ &= \{\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1 \mid [\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle] \\ &\qquad \in \ell_{\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1}(P\mathcal{H}_{A\in\Delta})\}.\end{aligned} \tag{3.16}$$

[22] The notation is potentially confusing here. The symbol $P\mathcal{H}$ denotes the projective Hilbert space—*i.e.*, the space of (complex) one-dimensional subspaces of $\mathcal{H}$; on the other hand, $P(\mathcal{H})$ denotes the space of projection operators on $\mathcal{H}$.

Note that if $\mathcal{H}_{A\in\Delta}$ is a one-dimensional subspace of $\mathcal{H}$, then Eq. (3.16) is equivalent to Eq. (3.14).

It must be emphasised that the assignment in Eq. (3.15) is intended to be *counterfactual*: we are not interested in state-vector reduction as it is normally understood, whether—as in the instrumentalist interpretation of quantum theory—it is regarded as a result of sub-ensemble selection, or whether—as in more adventurous interpretations—it is interpreted either as an effective physical process brought about by, for example, decoherence, or as an actual physical process associated with some non-linear modification of the Schrödinger equation. Rather, the intention is to assign the left ideal Eq. (3.15) (and similarly for Eq. (3.16)) in the monoid $\mathrm{Pr}P(\mathcal{H})$ as the truth value of the proposition "$A \in \Delta$" in the state $|\psi\rangle$ with the intent of producing a new type of 'neo-realist' interpretation of the quantum formalism: i.e., it is a non-standard (in the logical sense) way of saying 'how things are' in regard to the quantity A when the state is $|\psi\rangle$.

3.3. The Monoid of Strings of Projectors

At this point we should address a small defect in the formalism as presented so far. Namely, given a product $\hat{P}_n\hat{P}_{n-1}, \ldots, \hat{P}_1$ of projectors, it is not possible to recover the individual projectors from this operator since many different collections of projectors have the same product. In this sense, the statement above that Eq. (3.13) "assigns as a measure of the similarity between two state vectors the collections of those series of ideal measurements..." is not strictly correct, and the formalism must be modified slightly to gain the desired counterfactual interpretation of Eq. (3.13) and Eq. (3.15) (or Eq. (3.16)) in terms of strings of possible operations. This is done as follows.

The key idea is to construct a new monoid, $\mathrm{SP}(\mathcal{H})$, whose elements are finite strings of (non zero) projection operators, $R := (\hat{R}_p, \hat{R}_{p-1}, \ldots, \hat{R}_1)$ (p is called the *length* of the string) and with the monoid product law defined by concatenation of the strings. Thus if $R := (\hat{R}_p, \hat{R}_{p-1}, \ldots, \hat{R}_1)$ and $Q := (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_1)$ we define the product as

$$Q \star R := (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_1, \hat{R}_p, \hat{R}_{p-1}, \ldots, \hat{R}_1). \tag{3.17}$$

The unit element in the monoid $\mathrm{SP}(\mathcal{H})$ is the empty string, $\emptyset$. Physically, we think of the string $(\hat{R}_p, \hat{R}_{p-1}, \ldots, \hat{R}_1)$ as referring (counterfactually) to a situation in which the first operation corresponds to the projector $\hat{R}_1$, the second operation to $\hat{R}_2$, and so on.

If $R := (\hat{R}_p, \hat{R}_{p-1}, \ldots, \hat{R}_1)$ belongs to $\mathrm{SP}(\mathcal{H})$, we define the *reduction* of R to be the operator

$$\hat{R} := \hat{R}_p\hat{R}_{p-1}, \ldots, \hat{R}_1. \tag{3.18}$$

As a matter of convention, we define $\hat{\emptyset} := \hat{1}$, so that the unit element in the monoid $\mathrm{SP}(\mathcal{H})$ reduces to the unit operator. Note that $\widehat{Q \star R} = \hat{Q}\hat{R}$.[23]

We can now return to our ideas about generalised valuations in quantum theory and start by allowing the monoid $\mathrm{SP}(\mathcal{H})$ to act on $\mathcal{H}$ by

$$\ell_Q(|\psi\rangle) := \hat{Q}\,|\psi\rangle \tag{3.19}$$

for all finite strings Q of projectors. The expression Eq. (3.14) then gets replaced by

$$[\,[|\psi\rangle] = [|\phi\rangle]\,]_{B\mathrm{SP}(\mathcal{H})} := \{Q \in \mathrm{SP}(\mathcal{H}) \mid [\hat{Q}\,|\psi\rangle] = [\hat{Q}\,|\phi\rangle]\,\} \tag{3.20}$$

and the valuation in Eq. (3.16) becomes

$$V^{[|\psi\rangle]}(A \in \Delta)_{B\mathrm{SP}(\mathcal{H})} := \{Q \in \mathrm{SP}(\mathcal{H}) \mid [\hat{Q}\,|\psi\rangle] \in \ell_{\hat{Q}}(P\mathcal{H}_{A\in\Delta})\} \tag{3.21}$$

As desired, this is a left ideal in the monoid $\mathrm{SP}(\mathcal{H})$, and thereby gives a new generalised truth value for the proposition "$A \in \Delta$" in the quantum state $|\psi\rangle$.

3.4. The Question of Normalisation

If we think of a state of a quantum system as being determined by a normalised vector $|\psi\rangle$, then strictly speaking the state vector reduction should not be Eq. (3.11) but rather

$$|\psi\rangle \mapsto \frac{\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle}{\|\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle\|} \tag{3.22}$$

which is fine as long as $\|\hat{P}_n\hat{P}_{n-1},\ldots,\hat{P}_1\,|\psi\rangle\| \neq 0$. This is no problem in the conventional formalism since, there, one never gets reduction to an eigenstate for which there is *zero* probability of finding the associated eigenvalue. Or, more precisely: such zero probability events are swept under the carpet as never happening. However, for our neo-realist view, the normalisation problem is a genuine issue since in the action of the monoid $\mathrm{SP}(\mathcal{H})$ on a state $|\psi\rangle$, there will of course be strings Q for which $\hat{Q}\,|\psi\rangle = 0$.

There is an analogous normalisation issue for density matrices. In order to extend the formalism to include density-matrix states, we note first that the condition on the right hand side of the non-projective version of Eq. (3.21) would be $\hat{Q}\,|\psi\rangle \in \hat{Q}\mathcal{H}_{A\in\Delta}$, and this is equivalent to the statement that

$$\ell_Q(\hat{E}[A \in \Delta])\hat{Q}\,|\psi\rangle = \hat{Q}\,|\psi\rangle \tag{3.23}$$

where, in accordance with the remark following Eq. (3.3), $\ell_Q(\hat{E}[A \in \Delta])$ denotes the projection operator onto the subspace $\hat{Q}\mathcal{H}_{A\in\Delta}$. In turn, Eq. (3.23) is

[23] Note also that we allow consecutive repetition of projections operators in a string although, of course, the reduction of a string with a repeated projector is the same as that without.

equivalent to[24]

$$\langle\psi|\,\hat{Q}^{\dagger}\ell_{Q}(\hat{E}[A\in\Delta])\hat{Q}\,|\psi\rangle=\langle\psi|\,\hat{Q}^{\dagger}\hat{Q}\,|\psi\rangle. \tag{3.24}$$

Rewriting Eq. (3.23) in the form of Eq. (3.24) suggests how to extend the formalism to include states that are density matrices. We can define an action of the monoid $\mathrm{SP}(\mathcal{H})$ on the set of hermitian, trace-class operators $\hat{\rho}$ (i.e., the trace of ρ exists as a finite real number) by

$$\ell_{Q}(\hat{\rho}):=\hat{Q}\hat{\rho}\hat{Q}^{\dagger}. \tag{3.25}$$

Of course, if $\hat{\rho}$ is a density matrix state (so that $\mathrm{tr}(\hat{\rho})=1$) then we might want to define a normalised version of Eq. (3.25) as

$$\ell_{Q}(\hat{\rho}):=\frac{\hat{Q}\hat{\rho}\hat{Q}^{\dagger}}{\mathrm{tr}(\hat{Q}\hat{\rho}\hat{Q}^{\dagger})} \tag{3.26}$$

but this only makes sense if $\mathrm{tr}(\hat{Q}\hat{\rho}\hat{Q}^{\dagger})\neq 0$. However, we can avoid this difficulty by imitating Eq. (3.24) and defining the generalised valuation

$$V^{\hat{\rho}}(A\in\Delta)_{\mathrm{SP}(\mathcal{H})}:=\{Q\in\mathrm{SP}(\mathcal{H})\mid \mathrm{tr}(\hat{Q}\hat{\rho}\hat{Q}^{\dagger}\ell_{Q}(\hat{E}[A\in\Delta]))=\mathrm{tr}(\hat{Q}\hat{\rho}\hat{Q}^{\dagger})\}. \tag{3.27}$$

3.5. A New Category to Handle the Normalisation Issue

The trick used above to avoid the normalisation issue does not negate the fact that the right hand side of Eq. (3.21) (resp. Eq. (3.27)) necessarily includes strings Q for which $\hat{Q}\,|\psi\rangle=0$ (resp. $\hat{Q}\hat{\rho}\hat{Q}^{\dagger}=0$). Whether or not this is problematic is somewhat debatable. On the one hand, it is true that, as has been remarked already, in the conventional formalism such zeros do not occur. On the other hand, our monoid methods are aimed at giving a neo-realist interpretation of quantum theory, and, as such, it is not *a priori* necessary that they replicate exactly the structure of state-vector reduction in the conventional formalism. In that sense, the mathematics, as it is, does work.

However, if the normalisation question *is* thought to be a genuine issue, then the first step might well seem to be that we should restrict our attention to strings $Q:=(\hat{Q}_q,\hat{Q}_{q-1},\ldots,\hat{Q}_1)$ for which $\hat{Q}:=\hat{Q}_q\hat{Q}_{q-1},\ldots,\hat{Q}_1\neq 0$. We shall denote the set of all such strings by $\mathrm{SP}(\mathcal{H})_0$. Thus the elements of $\mathrm{SP}(\mathcal{H})_0$ have the property that they do not cause difficulties for *any* vectors in $\mathcal{H}$.

The problem, however, is that if Q_1 and Q_2 are members of $\mathrm{SP}(\mathcal{H})_0$, their monoid product Q_2Q_1 may not have this property. For example, considered as strings of unit length, any non-null projectors $\hat{P}$, $\hat{Q}$ belong to $\mathrm{SP}(\mathcal{H})_0$, but if $\hat{P}$ and $\hat{Q}$ are orthogonal then $\hat{Q}\hat{P}=0$.

[24] If $\hat{P}$ is any projector, and $|\phi\rangle$ is any vector, it follows from the Schwarz inequality that $\hat{P}\,|\phi\rangle=|\phi\rangle$ is equivalent to $\langle\phi|\,\hat{P}\,|\phi\rangle=\langle\phi|\phi\rangle$.

This means that $SP(\mathcal{H})_0$ is only a *partial* monoid, with the product $Q_2 Q_1$ being defined only if $\hat{Q}_2\hat{Q}_1 \neq 0$. There are several ways in which this problem can be tackled, and I will outline two of them here. A key observation is that a natural source of partial monoids is category theory, since the arrows in any category form a partial monoid: the composition $g \circ f$ of any two arrows f, g is only defined if the range of f is equal to the domain of g. This suggests trying to associate the elements of $SP(\mathcal{H})_0$ with the arrows in some category. One way is to define a new category $\mathcal{X}$ as follows:

(i) The objects are collections, Ξ, of non-zero vectors in $\mathcal{H}$ with the property that if $|\psi\rangle \in \Xi$, then, $\lambda\,|\psi\rangle \in \Xi$ for all $\lambda \in \mathbb{C}_*$.[25]
(ii) If Ξ_1 and Ξ_2 are a pair of objects, we define the arrows between them as the elements of the set

$$\mathrm{Hom}(\Xi_1, \Xi_2) := \{Q \in SP(\mathcal{H})_0 \mid \forall\,|\psi\rangle \in \Xi_1,\ \hat{Q}\,|\psi\rangle \in \Xi_2\} \tag{3.28}$$

$$\equiv \{Q \in SP(\mathcal{H})_0 \mid \hat{Q}\Xi_1 \subset \Xi_2\}. \tag{3.29}$$

If $Q \in \mathrm{Hom}(\Xi_1, \Xi_2)$ and $R \in \mathrm{Hom}(\Xi_2, \Xi_3)$ then the composite arrow $R \circ Q \in \mathrm{Hom}(\Xi_1, \Xi_3)$ is simply the concatenation of the strings.

Now, if $|\psi\rangle$, $|\phi\rangle$ belong to some object Ξ we can define, provisionally,

$$[\,|\psi\rangle = |\phi\rangle]_{\mathcal{X},\Xi} := \{Q \in \mathrm{Hom}(\Xi, \cdot) \mid \hat{Q}\,|\psi\rangle = \hat{Q}\,|\phi\rangle\} \tag{3.30}$$

where $\mathrm{Hom}(\Xi, \cdot)$ denotes the set of all arrows whose domain is Ξ. However, since the states concerned all have non-zero norm, it is better to replace Eq. (3.30) with the normalised form (and referring now to rays in the Hilbert space)

$$[[\,|\psi\rangle] = [\,|\phi\rangle]]_{\mathcal{X},\Xi} := \left\{ Q \in \mathrm{Hom}(\Xi, \cdot) \mid \exists z \in \mathbb{C}, |z| = 1, \frac{\hat{Q}\,|\psi\rangle}{\|\hat{Q}\,|\psi\rangle\|} = z\frac{\hat{Q}\,|\phi\rangle}{\|\hat{Q}\,|\phi\rangle\|} \right\} \tag{3.31}$$

which, of course, is not equivalent to Eq. (3.30) (we include the z phase factor since normalised states are only determined up to such factors). In fact, the condition on the right hand side of Eq. (3.31) is equivalent to the statement that there exists some $\lambda \in \mathbb{C}_*$ such that $\hat{Q}\,|\psi\rangle = \lambda\hat{Q}\,|\phi\rangle$.

It is clear that the right hand side of Eq. (3.31) is a sieve of arrows on Ξ, and hence a member of the Heyting algebra of all sieves on Ξ; as such it is a possible generalised truth value. It is easy to see how this would be extended to give generalised truth values to propositions "$|\psi\rangle \in K$" for a linear subspace

[25] It also would be possible to consider the objects to be subsets of rays. The analogue of our discussion for that case is obvious.

$K \subset \mathcal{H}$; in particular to subspaces of Ξ of the form $\mathcal{H}_{A\in\Delta}$. Namely, as:

$$V^{|\psi\rangle}(A \in \Delta)_{\mathcal{X},\Xi} := \{Q \in \mathrm{Hom}(\Xi, \cdot) \mid \hat{Q}\,|\psi\rangle \in \hat{Q}\mathcal{H}_{A\in\Delta}\}. \tag{3.32}$$

Note that if $\mathcal{H}_{A\in\Delta}$ is a one-dimensional subspace of $\mathcal{H}$, then Eq. (3.32) is equivalent to Eq. (3.31). Note also that Eq. (3.31) and Eq. (3.32) are 'contextual' in the sense that their right hand sides depend on the subset Ξ of vectors that is chosen to contain $|\psi\rangle$, as well as $\mathcal{H}_{A\in\Delta}$ of course.

To give more meaning to this construction we observe that there is an implicit 'polar operation' at play here, as encapsulated in the definition

$$\Xi^0 := \{Q \in \mathrm{SP}(\mathcal{H})_0 \mid \forall\,|\psi\rangle \in \Xi,\ \hat{Q}\,|\psi\rangle \neq 0\}. \tag{3.33}$$

Note that $\mathrm{Hom}(\Xi, \cdot) = \Xi^0$.

Similarly, if J is a subset of $\mathrm{SP}(\mathcal{H})_0$, we can define

$$J^0 := \{\,|\psi\rangle \in \mathcal{H}_* \mid \forall Q \in J,\ \hat{Q}\,|\psi\rangle \neq 0\} \tag{3.34}$$

where $\mathcal{H}_*$ denotes the set of all non-null vectors in $\mathcal{H}$. We note that $J_1 \subset J_2$ implies $J_2^0 \subset J_1^0$; similarly $\Xi_1 \subset \Xi_2$ implies $\Xi_2^0 \subset \Xi_1^0$. This is one reason for referring to these operations as 'polar'. Another is the fact that $\Xi_1^0 \cap \Xi_2^0 = (\Xi_1 \cup \Xi_2)^0$ for all objects Ξ_1 and Ξ_2, and similarly for pairs J_1 and J_2. We note that this construction can also be understood in the language of *Galois connections* (Bell, 1988) (which, in turn, are a special case of adjoint functors) defined on the partially ordered sets given by the subsets of $\mathrm{SP}(\mathcal{H})_0$ and the subsets of $\mathcal{H}_*$.[26]

We next note that

$$\begin{aligned}(\Xi^0)^0 &= \{\,|\psi\rangle \in \mathcal{H}_* \mid \forall Q \in \Xi^0,\ \hat{Q}\,|\psi\rangle \neq 0\}\\ &= \{\,|\psi\rangle \in \mathcal{H}_* \mid \forall\,|\phi\rangle \in \Xi,\ \hat{Q}\,|\phi\rangle \neq 0 \Rightarrow \hat{Q}\,|\psi\rangle \neq 0\}. \end{aligned} \tag{3.35}$$

In particular, $\Xi \subset (\Xi^0)^0$. In fact, $(\Xi^0)^0$ is a natural extension[27] of the subset of vectors Ξ in the sense that we can extend $\Xi \subset \mathcal{H}_*$ to $(\Xi^0)^0$ without changing the set of arrows with that particular domain. We will say that the subset Ξ is *full*[28] if $\Xi = (\Xi^0)^0$, and from now on we will write $(\Xi^0)^0$ as just Ξ^{00}. In a similar way, we can show that if $J \subset \mathrm{SP}(\mathcal{H})_0$ then $J \subset J^{00} := (J^0)^0$.

Now, for any subset of vectors $\Xi \subset \mathcal{H}_*$, we have $\Xi \subset \Xi^{00}$ and hence, in particular,

$$J^0 \subset (J^0)^{00} \tag{3.36}$$

[26] I thank Jeremy Butterfield for bringing this to my attention. For an application of the theory of Galois connections in standard quantum logic see Butterfield and Melia (1993).

[27] In the language of Galois connections, $(\Xi^0)^0$ is the 'closure' of Ξ.

[28] In the theory of Galois connections, it is standard to refer to such a set as 'closed'. However, this nomenclature is not used here to avoid confusion with topological closure.

for any subset $J \subset \mathrm{SP}(\mathcal{H})_0$. On the other hand, $J_1 \subset J_2$ implies $J_2^0 \subset J_1^0$; hence, in particular, the relation $J \subset J^{00}$ implies that

$$(J^{00})^0 \subset J^0. \tag{3.37}$$

Putting together Eqs. (3.36) and (3.37) we see that, for any subset $J \subset \mathrm{SP}(\mathcal{H})^0$

$$J^0 = (J^0)^{00}. \tag{3.38}$$

This means that it is easy to find subsets of non-null vectors that are full: namely, take the polar, J^0, of any subset J of $\mathrm{SP}(\mathcal{H})_0$ (conversely, any full subset, Ξ, of vectors is of the form J^0 for some J—just choose $J := \Xi^0$). In fact, it would be perfectly reasonable to require from the outset that the objects in our category $\mathcal{X}$ are only *full* subsets of vectors.

This is relevant to the remark that, although the approach above gives genuine generalised truth values of, for example, the type in Eq. (3.31), nevertheless there is no obvious physical significance of the 'context' in which such truth values arise: namely, the subset Ξ of non-null vectors in Eq. (3.31). However, if the objects are restricted to be full subsets of $\mathcal{H}$, and hence of the form J^0 for some $J \subset \mathrm{SP}(\mathcal{H})_0$, then the context is all those vectors that are 'reducible' with respect to the strings in J, which does have some physical content.

3.6. A Presheaf Approach to the Normalisation Problem

The basic problem of normalisation is encapsulated in the remark that if $\hat{P}$ is a projector such that $\hat{P}\,|\psi\rangle \neq 0$, then there will invariably be some projectors $\hat{Q}$ such that $\hat{Q}\hat{P}\,|\psi\rangle = 0$. The categorial approach in Section 3.5. is one way of enforcing the non-appearance of the undesired null vectors under multiplication of projection operators.

A somewhat different approach is based on the observation that although, for any given vector $|\psi\rangle$, $\hat{P}\,|\psi\rangle \neq 0$ does *not* imply $\hat{Q}\hat{P}\,|\psi\rangle \neq 0$, the equation $\hat{Q}\hat{P}\,|\psi\rangle \neq 0$ *does* imply that $\hat{P}\,|\psi\rangle \neq 0$. More generally, if we have a string $Q := (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_1)$ for which $\hat{Q}\,|\psi\rangle := \hat{Q}_q\hat{Q}_{q-1}, \ldots, \hat{Q}_1\,|\psi\rangle \neq 0$, then, necessarily, $\hat{Q}_{q-1}\hat{Q}_{q-2}, \ldots, \hat{Q}_1\,|\psi\rangle \neq 0$, $\hat{Q}_{q-2}\hat{Q}_{q-3}, \ldots, \hat{Q}_1\,|\psi\rangle \neq 0$ and so on. Thus although we cannot multiply projection operators at will, we *can* 'divide' by a projector in a string for which $|\psi\rangle$ is reducible. As we shall now see, this gives another way of handling the normalisation issue.

The first step is to observe that any monoid M gives rise to a category, $\tilde{M}$, whose objects $\tilde{M}$ are the elements of M, and whose arrows/morphisms are defined by[29]

$$\mathrm{Hom}(m_1, m_2) := \{m \in M \mid m_1 = m_2 m\}. \tag{3.39}$$

The identity arrow 1_m is defined as the unit element of M for all $m \in M$. Note that if $m : m_1 \to m_2$ (so that $m_1 = m_2 m$), and $m' : m_2 \to m_3$ (so that $m_2 = m_3 m'$) then $m_1 = m_2 m = m_3 m' m$ and hence the composition $m' \circ m : m_1 \to m_3$ must be defined as $m' \circ m := m'm$.

Although $\mathrm{SP}(\mathcal{H})_0$ is only a partial monoid, the general principle still holds, and we can construct the category $\widetilde{\mathrm{SP}(\mathcal{H})}_0$ whose objects are the elements of $\mathrm{SP}(\mathcal{H})_0$—*i.e.*, strings $Q := (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_1)$ for which $\hat{Q} := \hat{Q}_q \hat{Q}_{q-1}, \ldots, \hat{Q}_1 \neq 0$—and whose arrows are defined as

$$\mathrm{Hom}(Q_1, Q_2) := \{S \in \mathrm{SP}(\mathcal{H})_0 \mid Q_1 = Q_2 \star S\}. \tag{3.40}$$

Note that since the combination law in $\mathrm{SP}(\mathcal{H})_0$ is string concatenation, there is at most one arrow between any pair of objects. Hence this particular category is just a partially ordered set. Note also that if $S_1 \in \mathrm{Hom}(Q_1, Q_2)$ and $S_2 \in \mathrm{Hom}(Q_2, Q_3)$ then $Q_1 = Q_2 \star S_1$ and $Q_2 = Q_3 \star S_2$, so that $Q_1 = (Q_3 \star S_2) \star S_1 = Q_3 \star (S_2 \star S_1)$. Thus the arrow composition in this category is such that

$$S_2 \circ S_1 = S_2 \star S_1. \tag{3.41}$$

The empty string is a terminal object for $\widetilde{\mathrm{SP}(\mathcal{H})}_0$ since, for any object Q, we have $\mathrm{Hom}(Q, \emptyset) := \{S \in \mathrm{SP}(\mathcal{H})_0 \mid Q = S\} = \{Q\}$. Furthermore, if $Q := (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_1)$ is any object, the unique arrow $Q : Q \to \emptyset$ factors through a series of 'minimal' arrows that correspond to strings of unit length (*i.e.*, single projection operators):

$$\begin{aligned} &(\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_3, \hat{Q}_2, \hat{Q}_1) \xrightarrow{(\hat{Q}_1)} (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_3, \hat{Q}_2) \longrightarrow \\ &\xrightarrow{(\hat{Q}_2)} (\hat{Q}_q, \hat{Q}_{q-1}, \ldots, \hat{Q}_3) \longrightarrow \cdots \xrightarrow{(\hat{Q}_{q-1})} (\hat{Q}_q) \xrightarrow{(\hat{Q}_q)} \emptyset \end{aligned} \tag{3.42}$$

Now we discuss state-vector reduction in this context. This involves introducing the 'reduction presheaf' $\mathbf{R}$ on the category $\widetilde{\mathrm{SP}(\mathcal{H})}_0$. This is defined as follows:

[29] It is a matter of convention which way round the arrows are thought of as going. Thus it would be equally permissible to define $\mathrm{Hom}(m_2, m_1) := \{m \in M \mid m_1 = m_2 m\}$, and hence $\mathrm{Hom}(m_1, m_2) := \{m \in M \mid m_2 = m_1 m\}$, but we have chosen the definition in Eq. (3.39) as it is the most convenient one for the application we have in mind. Note that, in Eq. (3.39), an arrow $m : m_1 \to m_2$ means that m_2 is obtained from m_1 by 'right dividing' m_1 by m (not literally, of course, as m may not be invertible). With the alternative definition, an arrow $m : m_1 \to m_2$ means that m_2 is obtained from m_1 by right multiplying m_1 by m.

(i) To each object Q in the category $\mathrm{SP}\widetilde{(\mathcal{H})}_0$ we associate the space, $\mathbf{R}(Q)$, of vectors that are 'reducible' with respect to Q: *i.e.*, vectors $|\psi\rangle$ on which Q acts to give a reduction $\hat{Q}\,|\psi\rangle$ that is not the zero vector. Thus[30]

$$\mathbf{R}(Q) := \{\,|\psi\rangle \in \mathcal{H} \mid \hat{Q}\,|\psi\rangle \neq 0\} \tag{3.43}$$

(ii) If $S \in \mathrm{Hom}(Q_1, Q_2)$ is an arrow from Q_1 to Q_2 (so that $Q_1 = Q_2 \star S$) then we define the map $\mathbf{R}(S) : \mathbf{R}(Q_1) \to \mathbf{R}(Q_2)$ by

$$\mathbf{R}(S)\,|\psi\rangle := \hat{S}\,|\psi\rangle. \tag{3.44}$$

In regard to Eq. (3.44), note that if $|\psi\rangle \in \mathbf{R}(Q_1)$ then $\hat{Q}_1\,|\psi\rangle \neq 0$. However, $Q_1 = Q_2 \star S$ and hence $\hat{Q}_1 = \hat{Q}_2\hat{S}$, and thus $\hat{Q}_2\hat{S}\,|\psi\rangle \neq 0$. This means precisely that $\hat{S}\,|\psi\rangle \in \mathbf{R}(Q_2)$, and hence Eq. (3.44) does indeed define a map from $\mathbf{R}(Q_1)$ to $\mathbf{R}(Q_2)$.

Note that if $S_1 \in \mathrm{Hom}(Q_1, Q_2)$ and $S_2 \in \mathrm{Hom}(Q_2, Q_3)$ then $S_2 \circ S_1 \in \mathrm{Hom}(Q_1, Q_3)$ is defined by Eq. (3.41) as $S_2 \circ S_1 = S_2 \star S_1$ where, as we recall, '$\star$' denotes string concatenation. Then, if $|\psi\rangle \in \mathbf{R}(Q_1)$, we have

$$\begin{aligned}\mathbf{R}(S_2 \circ S_1)\,|\psi\rangle &= \mathbf{R}(S_2 \star S_1)\,|\psi\rangle = \widehat{S_2 \star S_1}\,|\psi\rangle = \hat{S}_2\hat{S}_1\,|\psi\rangle\\ &= \mathbf{R}(S_2)\mathbf{R}(S_1)\,|\psi\rangle\end{aligned} \tag{3.45}$$

so that $\mathbf{R}(S_2 \circ S_1) = \mathbf{R}(S_2)\mathbf{R}(S_1)$, as is required for a presheaf.

Note that, in regard to the chain of arrows in Eq. (3.42), the corresponding actions of the presheaf operators give the chain of reductions (c.f. Eq. (3.12))

$$|\psi\rangle \xrightarrow{\mathbf{R}(\hat{Q}_1)} \hat{Q}_1\,|\psi\rangle \xrightarrow{\mathbf{R}(\hat{Q}_2)} \hat{Q}_2\hat{Q}_1\,|\psi\rangle \cdots \xrightarrow{\mathbf{R}(\hat{Q}_q)} \hat{Q}_q\hat{Q}_{q-1}, \ldots, \hat{Q}_1\,|\psi\rangle. \tag{3.46}$$

This presheaf can be used to give a contextual, Heyting-algebra valued generalised truth structure. For example, if $|\psi\rangle$, $|\phi\rangle$ are a pair of vectors in $\mathbf{R}(Q)$ (so that $\hat{Q}\,|\psi\rangle \neq 0$ and $\hat{Q}\,|\phi\rangle \neq 0$), we provisionally define[31]

$$[\,|\psi\rangle = |\phi\rangle]_{\mathrm{SP}\widetilde{(\mathcal{H})}_0, Q} := \{S \in \mathrm{Hom}(Q, \cdot) \mid \hat{S}\,|\psi\rangle = \hat{S}\,|\phi\rangle\}. \tag{3.47}$$

Note that if $S \in \mathrm{Hom}(Q, \cdot)$ then $Q = Q' \star S$ for some string Q', and therefore, since $\hat{Q}\,|\psi\rangle \neq 0$ and $\hat{Q}\,|\phi\rangle \neq 0$, it follows that $\hat{S}\,|\psi\rangle \neq 0$ and $\hat{S}\,|\phi\rangle \neq 0$ in Eq. (3.48) (because $\widehat{Q' \star S} = \hat{Q}'\hat{S}$). We can therefore normalise the states and replace Eq. (3.47) with (and referring now to rays in the Hilbert

[30] Equivalently, we could define $\mathbf{R}(Q)$ to be the set of all *rays* in $\mathcal{H}$ that are not annihilated by $\hat{Q}$.

[31] From a topos perspective, Eq. (3.48) is the characteristic arrow $\mathrm{eq}_{\mathbf{R}} : \mathbf{R} \times \mathbf{R} \to \boldsymbol{\Omega}$ of the diagonal subobject $\triangle : \mathbf{R} \to \mathbf{R} \times \mathbf{R}$. Here, $\boldsymbol{\Omega}$ denotes the presheaf of sieves on the category $\mathrm{SP}(\mathcal{H})_0$.

space)

$$[[\,|\psi\rangle] = [\,|\phi\rangle]]_{\mathrm{SP}(\widetilde{\mathcal{H}})_0,Q} := \left\{ S \in \mathrm{Hom}(Q,\cdot) \mid \exists z \in \mathbb{C}, |z| = 1, \frac{\hat{S}\,|\psi\rangle}{\|\hat{S}\,|\psi\rangle} = z\frac{\hat{S}\,|\phi\rangle}{\|\hat{S}\,|\phi\rangle} \right\} \tag{3.48}$$

which, of course, is not equivalent to Eq. (3.47).

The right hand side of Eq. (3.48) is *contextual* in the sense that it depends on the object Q that is chosen at which to affirm the statement "$[\,|\psi\rangle] = [\,|\phi\rangle]$". There could be other spaces $\mathbf{R}(Q')$ to which both $|\psi\rangle$ and $|\phi\rangle$ belong, and the truth value in the context Q', namely $[[\,|\psi\rangle] = [\,|\phi\rangle]]_{\mathrm{SP}(\widetilde{\mathcal{H}})_0,Q'}$, would not be the same as that in the context Q.

The logical structure of these contextual truth values arises because the right hand side of Eq. (3.48) is a *sieve* of arrows on Q, and hence an element of the Heyting algebra of all such sieves on Q. This is how generalised truth values arise in the present approach. Note that if $K \subset \mathcal{H}$ is a subset of vectors, all of which are Q-reducible (so that none of them are annihilated by $\hat{Q}$) then we can define the valuation

$$[\,|\psi\rangle \in K]_{\mathrm{SP}(\widetilde{\mathcal{H}})_0,Q} := \{S \in \mathrm{Hom}(Q,\cdot) \mid \hat{S}\,|\psi\rangle \in \hat{S}K\} \tag{3.49}$$

which gives a contextual, generalised measure of the extent to which the vector $|\psi\rangle$ (viewed as a member of $\mathbf{R}(Q)$), resp. the associated ray $[\,|\psi\rangle]$, belongs to, resp is a subspace of, the subspace $K \subset \mathbf{R}(Q)$. In particular, if $K := \mathcal{H}_{A\in\Delta}$, we arrive at the generalised valuation[32]

$$V^{[\,|\psi\rangle]}(A \in \Delta)_{\mathrm{SP}(\widetilde{\mathcal{H}})_0,Q} := \{S \in \mathrm{Hom}(Q,\cdot) \mid \hat{S}\,|\psi\rangle \in \hat{S}\mathcal{H}_{A\in\Delta}\} \tag{3.50}$$

which is a sieve at Q. We thereby obtain a new candidate for a generalised truth value for the proposition "$A \in \Delta$" in the context Q when the state is $|\psi\rangle$ (or, equivalently, the ray $[\,|\psi\rangle]$).

4. CONCLUSION

This paper is a contribution to the long-standing question of whether the standard quantum formalism can be given an interpretation that does not involve measurement as a fundamental category. This is essential in quantum cosmology, and it is a very non-trivial problem. Of course, it is quite possible that the quantum formalism itself needs changing in the cosmological context, but the working

[32] From a topos perspective, the right hand side of Eq. (3.49) is the 'evaluation arrow' $\mathrm{eval}_{\mathbf{R}} : \mathbf{R} \times P\mathbf{R} \to \mathbf{\Omega}$. This is the topos equivalent of the fact that, in normal set theory, if $J \subset X$ and if $x \in X$, the pair $(x, J) \in X \times PX$ can be mapped to the value $1 \in \{0, 1\}$ if $x \in J$, and to $0 \in \{0, 1\}$ if $x \notin J$.

assumption here is that this is not the case, and that we must therefore strive to give a 'neo-realist' interpretation to standard quantum theory.

In the earlier series of papers by the author and collaborators it was shown how topos theory could be used to give a generalised truth value to the propositions in a quantum theory. The topos concerned involved presheaves over a variety of different categories, including the category of self-adjoint operators, the category of Boolean subalgebras of the lattice of projectors, and the category of abelian von Neumann algebras.

In the present paper we have concentrated instead on the uses of the topos of M-sets for various monoids M. We showed that our earlier results in classical physics can be recovered using the monoid $C^\infty(\mathbb{R}, \mathbb{R})$, and that our earlier results in quantum physics can be recovered using the monoid $M(\mathbb{R}, \mathbb{R})$.

Then we considered possible applications of the monoid $L(\mathcal{H})$ of all bounded operators on the Hilbert space $\mathcal{H}$ of the quantum theory. This led rather naturally to thinking about the monoid of strings of projection operators, and hence ultimately to the production of a new generalised valuation in quantum theory whose truth values are determined by what would be state-vector reductions in the standard instrumentalist interpretation.

If we are not worried about the normalisation issue, then the final result is Eq. (3.21) (or Eq. (3.27) for a density matrix state $\hat{\rho}$). This is a *bona fide* alternative to the valuation Eq. (2.22) of our earlier work. If the normalisation problem is of concern, then more sophisticated ideas are needed, two of which are discussed in the present paper. This leads to the generalised valuations in Eqs. (3.32) and (3.50) whose values lie in sieves over the chosen context/object Ξ and Q respectively. These results have obvious extensions to the situation where the state is a density matrix.

It should be emphasised that the material in the present paper represents only a preliminary investigation of the application of M-sets to quantum theory, and much work remains to be done. In particular, it is important to see to what extent the probabilistic predictions in standard quantum theory can be *recovered* from the logical values of the generalised valuations we have discussed above. Ideally, one would like to recover *all* the standard probabilistic predictions, so that the logical structure alone is sufficient to encapsulate the generalised ontology that is inherent in neo-realist interpretations of the present type. Hopefully, this will be the subject of a later paper.

Another potential application of the monoid of strings of projectors is to consistent history theory in which products of projectors play a fundamental role (Griffiths, 1984; Omnès, 1988; Gell-Mann and Hartle, 1990; Isham, 1994); one early attempt to discuss consistent history theory in topos language is Isham (1997). There are also strong links to the much earlier work on Baer-$*$ rings in quantum logic (Pool, 1975) as well as work on the use of Galois connections in quantum theory (Butterfield and Melia, 1993).

On the other hand, whilst I was completing this paper, a preprint appeared very recently with interesting overlaps with some of the ideas above (Lehmann *et al.*, 2006). This paper deals with an abstract 'algebra of measurements' whose basic ingredient is a monoid of functions from a space X to itself. In particular, what these authors call 'cumulativity' is related to the ideas above about using left ideals in LM, or sieves. In general, this interesting approach can clearly be integrated into the discussion of the present paper. These topics all deserve further study.

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